MA 3053 Practice Exam 1

1.) pf: S'pose $f(x)=f(y)$ for some $x,y \in X$. Then $f(f(x))=f(f(y)) \Rightarrow (f\circ f)(x)=(f\circ f)(y) \Rightarrow x=y$. $\therefore f$ is 1-1. Now for any $y \in X$ consider $y=f(x)$. Then $f(y)=f(f(x))=(f\circ f)(x)=x$. $\therefore x=f(y)\in X$ since f is a fnc. $\therefore f$ is onto. Thus f is a bijection. $\square$

2.) pf: f, g must both be injections. S'pose $f(a_1)=f(a_2)$ for $a_1,a_2 \in A$ and $g(b_1)=g(b_2)$ for $b_1,b_2\in B$ then by def of $C\times D$ $(f(a_1),g(b_1))=(f(a_2),g(b_2)) \Rightarrow h(a_1,b_1)=h(a_2,b_2)$. Since h is 1-1 $\Rightarrow a_1=a_2$ and $b_1=b_2$ $\therefore f$ and g are both 1-1. $\square$

3.) pf:

(1) let $(x,y)\in(A\cap B)\times(C\cap D) \Rightarrow x\in A\cap B$ and $y\in C\cap D$ $\therefore x\in A$ and $x\in B$ and $y\in C$ and $y\in D$. So $(x,y)\in A\times C$ and $(x,y)\in B\times D$ $\therefore (x,y)\in(A\times C)\cap(B\times D)$. Since (x,y) arb. $\Rightarrow (A\cap B)\times(C\cap D)\subseteq(A\times C)\cap(B\times D)$

(2) let $(x,y)\in(A\times C)\cap(B\times D) \Rightarrow$ ~~$x\in A$~~ $(x,y)\in A\times C$ and $(x,y)\in B\times D$ $\therefore x\in A$ and $y\in C$ and $x\in B$ and $y\in D$. So $x\in A\cap B$ and $y\in C\cap D$. $\therefore (x,y)\in(A\cap B)\times(C\cap D)$. Since (x,y) arb $\Rightarrow (A\times C)\cap(B\times D)\subseteq(A\cap B)\times(C\cap D)$.

$\therefore$ by (1) and (2) $(A\cap B)\times(C\cap D)=(A\times C)\cap(B\times D)$. $\square$

4.) pf:

(1) let $f(x)\in f(\bigcap_{\alpha\in A}P_\alpha) \Rightarrow x\in\bigcap_{\alpha\in A}P_\alpha \Rightarrow x\in P_\alpha$ for all $\alpha\in A$. $\Rightarrow f(x)\in f(P_\alpha)$ for all $\alpha\in A \Rightarrow f(x)\in\bigcap_{\alpha\in A}f(P_\alpha)$. Since $f(x)$ was arb. $\Rightarrow f(\bigcap_{\alpha\in A}P_\alpha)\subseteq\bigcap_{\alpha\in A}f(P_\alpha)$.

(2) let $f(x)\in\bigcap_{\alpha\in A}f(P_\alpha)$. $\Rightarrow f(x)\in f(P_\alpha)$ for all $\alpha\in A$. Since f is 1-1 $\Rightarrow x\in P_\alpha$ for all $\alpha\in A \Rightarrow x\in\bigcap_{\alpha\in A}P_\alpha$ $\therefore f(x)\in f(\bigcap_{\alpha\in A}P_\alpha)$. Since $f(x)$ was arb. $\Rightarrow \bigcap_{\alpha\in A}f(P_\alpha)\subseteq f(\bigcap_{\alpha\in A}P_\alpha)$.

$\therefore$ by (1) and (2) $f(\bigcap_{\alpha\in A}P_\alpha)=\bigcap_{\alpha\in A}f(P_\alpha)$. $\square$

5.) pf: refl: Does there exist a $z\in\mathbb{N}^+$ s.t. $xz=x$? yes pick $z=1$. antisym: let $x\preceq y$ and $y\preceq x$. Then there are $z_1,z_2\in\mathbb{N}^+$ s.t. $xz_1=y$ and $yz_2=x$. So $xz_1z_2=yz_2=x \Rightarrow z_1z_2=1$. Since $z_1,z_2\in\mathbb{N}^+ \Rightarrow z_1=z_2=1$. $\therefore x=y$.

trans: let $x\preceq y$ and $y\preceq z$, then there are $a,b\in\mathbb{N}^+$ s.t. $xa=y$ and $yb=z$. $\Rightarrow xab=yb=z$. So $x(ab)=z$. Since $a,b\in\mathbb{N}^+\Rightarrow ab\in\mathbb{N}^+$ so $x\preceq z$. $\therefore \preceq$ is a partial order. $\square$.

6.) pf: refl: is it true $f\preceq f$? by def of $\preceq$, $f(x)\le f(x)$ which is true so, yes. antisym: let $f\preceq g$ and $g\preceq f$. Then $f(x)\le g(x)$ and $g(x)\le f(x)$ for all $x\in X$. $\Rightarrow f(x)=g(x)$ for all $x\in X$. $\therefore f=g$.

trans: let $f\preceq g$ and $g\preceq h$. Then $f(x)\le g(x)$ and $g(x)\le h(x)$ for all $x\in X$. $\Rightarrow f(x)\le g(x)\le h(x)$ for all $x\in X$ so $f(x)\le h(x)$ for all $x\in X$ $\therefore f\preceq h$. So $\preceq$ is a partial order. $\square$

7.) pf:
ref: by def of $\sim$, consider $f(x)=f(x)$ which is true. $\therefore x\sim x$. ~~Sym~~: let $x\sim y$. $\therefore f(x)=f(y)$
so $f(y)=f(x) \Rightarrow y\sim x$. trans: let $x\sim y$ and $y\sim z$. Then $f(x)=f(y)$ and $f(y)=f(z)$.
so $f(x)=f(y)=f(z) \Rightarrow f(x)=f(z)$ so $x\sim z$. $\therefore$ $\sim$ is an equiv. rel. By def $[x]=\{y\in X : x\sim y\}$
$\therefore [x]=\{y\in X : f(x)=f(y)\}$ so if $b\in Rn(f)$ and ~~...~~ $f(y)=b \Rightarrow y=f^{-1}(b)$
$\therefore [x]=\{f^{-1}(b) : \underset{\text{fixed}}{b\in Rn(f)}\}$ so $X/\sim \; = \{f^{-1}(b) : b\in Rn(f)\}$. $\square$

8.) pf:
ref: is it true $(a,b)\sim(a,b)$? by def of $\sim$ need $a+b=b+a$, but since add. is commutative
this is true. sym: let $(a,b)\sim(c,d) \Rightarrow a+d=b+c \Rightarrow b+c=a+d$ so $c+b=d+a$
so $(c,d)\sim(a,b)$. Trans: let $(a,b)\sim(c,d)$ and $(c,d)\sim(x,y)$. So $a+d=b+c$ and $c+y=d+x$
$\Rightarrow a+d+y=b+c+y=b+d+x$ since add. comm. $\Rightarrow a+y+d=b+x+d \Rightarrow a+y=b+x$.
$\therefore (a,b)\sim(x,y)$. $\therefore$ $\sim$ is an equiv. rel. $\square$

9.) pf:
Since both $A\subseteq C$ and $B\subseteq A$. Then if $x\in A\cup B \Rightarrow x\in A$ or $x\in B$, by assumption $B\subseteq A\subseteq C$ $\therefore x\in C$
so $x\in A\cap C$. Since x arb. $\Rightarrow A\cup B\subseteq A\cap C$. OTH if $x\in A\cap C \Rightarrow x\in A$ and $x\in C$.
But if $x\in A$ then $x\in A\cup B$ by def. since x arb. $\Rightarrow A\cap C\subseteq A\cup B$. $\therefore A\cap C=A\cup B$ which
is a contradiction. $\square$